Application of Structural Physical Approximation to Partial Transpose in Teleportation

Satyabrata Adhikari
Delhi Technological University (DTU)

Singlet fraction and its usefulness in Teleportation

- Singlet fraction: $F(\sigma) = \max_{|\psi_{ME}\rangle} \langle \psi_{ME} | \sigma | \psi_{ME} \rangle$

- Teleportation fidelity: $f_T(\sigma) = \dfrac{2F(\sigma)+1}{3}$

- A mixed two-qubit entangled state useful for teleportation if the singlet fraction is greater than ½.

M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999).

Results

- Proved that the optimal trace preserving protocol for maximizing the singlet fraction of a given state always belongs to a class of one-way communication (1-LOCC).
- Shown that any entangled two-qubit mixed state can be used as a resource for quantum teleportation using certain trace preserving local operations and classical communications.

F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Result

- Convex optimization problem:

$$F^*(\rho_{12}) = \underset{X}{Max}[\frac{1}{2} - Tr(X\rho_{12}^{T_B})]$$

$$Subject\ to\quad 0 \le X \le I_4$$

$$\frac{-I_4}{2} \le X \le \frac{I_4}{2}$$

$X = (A \otimes I_2)\left|\psi^-\right\rangle\left\langle\psi^-\right|(A^+ \otimes I_2)$, *and* A represents the filter.

$$\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

$$F^*(\rho_{12}) = \frac{1}{2} - Tr(X^{opt}\rho_{12}^{T_B})$$

F. Verstraete and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)

Motivations

1. In the expression of optimal singlet fraction, we have $\rho_{12}^{T_B}$, which is an unphysical operation and cannot be implemented in a lab.

2. Singlet fraction is not normalized and thus for some entangled state, it may take value greater than unity. Important point is that the filter and hence singlet fraction depends on the state under investigation.

$$\rho_{12}=\begin{bmatrix}0&0&0&0\\0&\frac{1}{2}&1&0\\0&1&\frac{1}{2}&0\\0&0&0&0\end{bmatrix}\quad \rho_{12}^{T_B}=\begin{bmatrix}0&0&0&1\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{2}&0\\1&0&0&0\end{bmatrix}\qquad X=\begin{bmatrix}\frac{a^2}{2}&0&0&\frac{a}{2}\\0&0&0&0\\0&0&0&0\\\frac{a}{2}&0&0&\frac{1}{2}\end{bmatrix},\quad -1\le a\le 1$$

$$Tr(X\rho_{12}^{T_B})=a \qquad F^*(\rho_{12})=\frac{1}{2}-a$$

Positive Map

- A linear map $\Lambda : B(H_A) \rightarrow B(H_B)$ is called positive if for all positive $X \in B(H_A)$, the operator $\Lambda(X) \in B(H_B)$ is positive.
- From the definition, it is clear that positive maps applied to density matrices give density matrices.
- Transposition map is an example of a positive map.
- Now the question arises that whether all positive maps applied to density matrices in the extended Hilbert space give density matrices?

Completely Positive Map (CP)

- Let $\Lambda : B(H_A) \to B(H_B)$ be a positive map and let $I_d : M_d(C) \to M_d(C)$ denote an identity map. Then we say that Λ is completely positive if for all d, the extended map $I_d \otimes \Lambda$ is positive.
- Trace map is an example of a completely positive map.
- CP represent the most general transformations of quantum states. **[K. Kraus, States, Effects, and Operations (SpringerVerlag, Berlin, 1983)].**

Why CP Map is important?

- CP maps are capable to describe an arbitrary quantum transmission channel. **[B. Schumacher, Phys. Rev. A 54, 2614 (1996)]**
- Some unphysical transformations, such as quantum cloners or universal-NOT gate can be approximated optimally by an optimal CP map. **[J. Fiurasek, Phys. Rev. A 64, 062310 (2001)].**

Positive but not Completely Positive Map

- Transposition map is a positive but not completely positive map because it is not 2-positive.
- Positive but not completely positive maps are very useful to detect entangled states.
- A density matrix of a bipartite system is inseparable iff there exists a positive map acting on a subsystem such that the image of the density matrix is not positive semidefinite. **[M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).]**

Positive but not Completely Positive Map

- Particularly important is the partial transposition map $P = I_A \otimes T_B$, whose ability to detect the entanglement was first pointed out by Peres **[A. Peres, Phys. Rev. Lett. 77, 1413 (1996)].**
- Since P is not a CP map, it seems to be impossible to implement this map physically in a lab.
- To overcome this difficulty, Horodecki and Ekert suggested a way to approximate the unphysical map such that it can be implemented in experiment. This method is called Structural Physical Approximation. **[P. Horoecki and A. Ekert, Phys. Rev. Lett. 89, 127902-1(2002)]**

Structural Physical Approximation (SPA)

- SPA is a transformation from positive maps to completely positive maps.
- SPA has been exploited to approximate unphysical operation such as partial transpose.
- The idea is to form a mixture of the positive map P with a CP map O that transforms all quantum states onto maximally mixed state.
- It can be used directly in experiment to detect the entanglement directly without full tomographic reconstruction of the bipartite state whose entanglement is to be determined.

SPA to Partial Transposition

Any arbitrary two qubit density matrix is given by

$$\rho_{12} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{12}^* & t_{22} & t_{23} & t_{24} \\ t_{13}^* & t_{23}^* & t_{33} & t_{34} \\ t_{14}^* & t_{24}^* & t_{34}^* & t_{44} \end{bmatrix}$$

where (*) denotes the complex conjugate.

The structural physical approximation of partial transposition of ρ_{12} is given by

$$\tilde{\rho}_{12} = \left[\frac{1}{3}(I \otimes \tilde{T}) + \frac{2}{3}(\tilde{\theta} \otimes D)\right] \rho_{12}$$

D: Depolarization

$$D[\rho] = \frac{I_2}{2}$$

D can be performed by random application of Pauli matrices:

$$D(.) = \frac{1}{4} \sum_{i=0,x,y,z} \sigma_i (.) \sigma_i$$

H-T.Lim et.al., Phys. Rev. Lett. 107, 160401 (2011)

SPA to Partial Transposition

$\tilde{T}$ and $\tilde{\theta}$: SPA to transpose operation and inversion operation.

$$\tilde{T}(\rho) = \sum_{k=1}^{4} Tr(M_k \rho)\left|v_k\right\rangle\left\langle v_k\right|$$

$$where,\ \left|v_1\right\rangle \propto \left|0\right\rangle + \frac{ie^{i\pi\frac{2}{3}}}{i+e^{-i\pi\frac{2}{3}}}\left|1\right\rangle,\ \left|v_2\right\rangle \propto \left|0\right\rangle - \frac{ie^{i\pi\frac{2}{3}}}{i-e^{-i\pi\frac{2}{3}}}\left|1\right\rangle$$

$$\left|v_3\right\rangle \propto \left|0\right\rangle + \frac{ie^{i\pi\frac{2}{3}}}{i-e^{-i\pi\frac{2}{3}}}\left|1\right\rangle, \left|v_4\right\rangle \propto \left|0\right\rangle - \frac{ie^{i\pi\frac{2}{3}}}{i+e^{-i\pi\frac{2}{3}}}\left|1\right\rangle$$

$$And \quad M_k = \frac{1}{2}\left|v_k^*\right\rangle\left\langle v_k^*\right|$$

$$\tilde{\theta}(.) = \sigma_y \tilde{T}(.)\sigma_y,\ where\ \sigma_y = -i\left|0\right\rangle\left\langle 1\right| + i\left|1\right\rangle\left\langle 0\right|$$

Matrix form of $\tilde{\rho}_{12}$

$$\tilde{\rho}_{12} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12}^{*} & E_{22} & E_{23} & E_{24} \\ E_{13}^{*} & E_{23}^{*} & E_{33} & E_{34} \\ E_{14}^{*} & E_{24}^{*} & E_{34}^{*} & E_{44} \end{bmatrix}$$

$$where\ \ E_{11} = \frac{1}{9}(2+t_{11}), E_{12} = \frac{1}{9}(-it_{12}+t_{12}^{*}), E_{13} = \frac{1}{9}(t_{13}-i\,(t_{13}^{*}+t_{24}^{*})), E_{14} = \frac{1}{9}(-it_{14}+t_{23})$$

$$E_{22} = \frac{1}{9}(2+t_{22}),\ E_{23} = \frac{1}{9}(t_{14}+it_{23}),\ E_{24} = \frac{-i}{9}(t_{13}^{*}+t_{24}^{*}),$$

$$E_{33} = \frac{1}{9}(2+t_{33}), E_{34} = \frac{1}{9}(-it_{34}+t_{34}^{*})$$

$$E_{44} = \frac{1}{9}(2+t_{44})$$

How does SPA works?

Let us consider a two qubit state described by the density matrix

$$\rho_{12} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c^* & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix}, \quad a+b+d+e=1$$

Partial transpose of ρ_{12} is given by

$$\rho_{12}^{T_B} = \begin{bmatrix} a & 0 & 0 & c \\ 0 & b & 0 & 0 \\ 0 & 0 & d & 0 \\ c^* & 0 & 0 & e \end{bmatrix}$$

SPA to partial transpose of ρ_{12} is given by

$$\tilde{\rho}_{12} = \begin{bmatrix} \frac{1}{9}(2+a) & 0 & 0 & \frac{c}{9} \\ 0 & \frac{1}{9}(2+b) & \frac{ic}{9} & 0 \\ 0 & \frac{-ic^*}{9} & \frac{1}{9}(2+d) & 0 \\ \frac{c^*}{9} & 0 & 0 & \frac{1}{9}(2+e) \end{bmatrix}$$

Condition of entanglement in terms of eigenvalues of $\tilde{\rho}_{12}$

Let us consider the operator: $O = \tilde{\rho}_{12} - \rho_{12}^{T_B}$

The expectation value of the operator *O* in the state $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$ is given by

$$\langle\phi|O|\phi\rangle = \frac{2}{9} \Rightarrow Tr[(\tilde{\rho}_{12} - \rho_{12}^{T_B})|\phi\rangle\langle\phi|] = \frac{2}{9}$$

$$\Rightarrow Tr[W\rho_{12}] = Tr[\tilde{\rho}_{12}|\phi\rangle\langle\phi|] - \frac{2}{9}, \; where \; W = |\phi\rangle^{T_B}\langle\phi|$$

Let $\lambda_{\min}$ be the minimum eigenvalue of $\tilde{\rho}_{12}$ and

$|\phi\rangle$ be the corresponding eigenvector, then

$$\tilde{\rho}_{12}|\phi\rangle = \lambda_{\min}|\phi\rangle$$

Therefore, $Tr[W\rho_{12}] = \lambda_{\min} - \frac{2}{9}$

Condition of entanglement in terms of eigenvalues of $\tilde{\rho}_{12}$

If the state ρ_{12} is entangled and ***W*** detects it, then

$$Tr[W\rho_{12}] < 0 \Rightarrow \lambda_{\min} < \frac{2}{9}, \quad where\ W = |\phi\rangle^{T_B}\langle\phi|$$

Horodecki and Ekert, Phys. Rev. Lett. 89, 127902 (2002)

Is there exist any hermitian operator that detect whether the eigenvalue of $\tilde{\rho}_{12}$ is less than $\frac{2}{9}$?

Construction of the Hermitian operator

Recall $Tr[W\rho_{12}] = Tr[\tilde{\rho}_{12}|\phi\rangle\langle\phi|] - \frac{2}{9}$ $= Tr[(|\phi\rangle\langle\phi| - \frac{2}{9}I)\tilde{\rho}_{12}]$

Again, if the state ρ_{12} is entangled then

$$Tr[W\rho_{12}] < 0,\ where\ W = |\phi\rangle^{T_B}\langle\phi|$$

Equivalent, we can say that $Tr[(|\phi\rangle\langle\phi| - \frac{2}{9}I)\tilde{\rho}_{12}] < 0$

The above condition implies that the operator $|\phi\rangle\langle\phi| - \frac{2}{9}I$

detect that the eigenvalue of $\tilde{\rho}_{12}$ is less than $\frac{2}{9}$

Properties of the operator $|\phi\rangle\langle\phi|-\frac{2}{9}I$

P1. The operator is Hermitian and has eigenvalues $\frac{7}{9},\frac{-2}{9},\frac{-2}{9},\frac{-2}{9}$

thus the hermitian operator has at least one negative eigenvalue.

P2. For all separable state ρ_{12}^{S} , it can be easily shown that the operator detect that the eigenvalue of $\tilde{\rho}_{12}$ is greater equal to $\frac{2}{9}$

P3. The operator can be converted into a normalized operator by multiplying with a suitable constant

Hence the above defined operator satisfies all the properties of a witness operator. This witness operator detect whether the eigenvalue of $\tilde{\rho}_{12}$ is less than $\frac{2}{9}$ or not? Equivalently, whether the state ρ_{12} is entangled or not?

Teleportation with SPA to PT

Let us consider the filter *A* of this form

$$A=\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix},\ -1\le a\le 1$$

Then X is given by

$$X=\begin{pmatrix} \frac{a^2}{2} & 0 & 0 & \frac{a}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Let us consider the operator $X(\tilde{\rho}_{12}-\frac{1}{9}\rho_{12}^{T_B}),$ and calculate the trace of it.

After simple algebra, we get

$$Tr[X(\tilde{\rho}_{12} - \frac{1}{9}\rho_{12}^{T_B})] = \frac{2}{9}(a^2+1)$$

$$\frac{1}{2} - Tr(X\rho_{12}^{T_B}) = \frac{1}{2} - (9Tr(X\tilde{\rho}_{12}) - 2(a^2+1))$$

$$\Rightarrow F(\rho_{12}) = \frac{1}{2} - (9Tr(X\tilde{\rho}_{12}) - 2(a^2+1))$$

Now we have to search for X for which $F(\rho_{12}) > \frac{1}{2}$.

$F(\rho_{12}) > \frac{1}{2}$ holds only when $9Tr(X\tilde{\rho}_{12}) - 2(a^2+1) < 0$.

The above condition implies

$$Tr\left[\left(\frac{1}{a^2+1}X\right)\tilde{\rho}_{12}\right] < \frac{2}{9}$$

Recall (i) $Tr[(|\phi\rangle\langle\phi| - \frac{2}{9}I)\tilde{\rho}_{12}] < 0, \; where \;\; |\phi\rangle = \alpha|00\rangle + \beta|11\rangle$

and (*ii*) $\tilde{\rho}_{12}|\phi\rangle = \lambda_{\min}|\phi\rangle$, *where* $\lambda_{\min}$ is the eigenvalue and $|\phi\rangle$ is the corresponding eigenvector.

In $\{\,|00\rangle, |11\rangle\,\}$ subspace, $\frac{1}{a^2+1}X$ can be expressed as

$$\frac{1}{a^2+1}X = |\chi\rangle\langle\chi|, \; where \; |\chi\rangle = \frac{1}{\sqrt{a^2+1}}\left(a|00\rangle + |11\rangle\right)$$

Therefore, $|\phi\rangle$ and $|\chi\rangle$ are parallel vectors and hence there exist a real scalar k such that

$$|\chi\rangle = k|\phi\rangle.$$

It can be easily shown that $|\chi\rangle$.is an eigenvector of $\tilde{\rho}_{12}$ for the eigenvalue $\lambda_{\min}$.

Then $Tr\left[\left(\frac{1}{a^2+1}X\right)\tilde{\rho}_{12}\right] < \frac{2}{9} \Rightarrow Tr\left[|\chi\rangle\langle\chi|\tilde{\rho}_{12}\right] < \frac{2}{9}$

$$\Rightarrow \lambda_{\min} < \frac{2}{9}$$

Singlet fraction in terms of eigenvalue $\lambda_{\min}$ of $\tilde{\rho}_{12}$ is given by

$$F(\rho_{12}) = \frac{1}{2} - \frac{9}{a^2+1}(\lambda_{\min} - \frac{2}{9})$$

We found that if we restrict the value of $\lambda_{\min}$ in the interval $\left[\frac{1}{6}, \frac{2}{9}\right)$, then the range of singlet fraction is given by

$$\frac{1}{2} < F(\rho_{12}) \leq \frac{1}{2} + \frac{1}{2(a^2+1)}$$

Optimal singlet fraction in terms of minimum eigenvalue of $\tilde{\rho}_{12}$

Optimal singlet fraction $F^{opt}(\rho_{12})$ can be achieved by putting $a = 0$ in

$$F(\rho_{12}) = \frac{1}{2} - \frac{9}{a^2+1}(\lambda_{\min} - \frac{2}{9})$$

The optimal singlet fraction is given by

$$F^{opt}(\rho_{12}) = \frac{1}{2} - 9\left[\lambda_{\min} - \frac{2}{9}\right], \quad \frac{1}{6} \le \lambda_{\min} < \frac{2}{9}$$

This is the required condition in terms of eigenvalues of $\tilde{\rho}_{12}$, for ρ_{12} to be used as a resource state for quantum teleportation

Example

The density matrix corresponding to $\left|\phi^+\right\rangle_{12}$ is given by

$$\rho_{12} = \left|\phi^+\right\rangle_{12}\left\langle\phi^+\right| = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad \textit{where} \quad \left|\phi^+\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

SPA to partial transpose of ρ_{12} is given by

$$\tilde{\rho}_{12} = \begin{bmatrix} \frac{5}{18} & 0 & 0 & \frac{-i}{18} \\ 0 & \frac{2}{9} & \frac{1}{18} & 0 \\ 0 & \frac{1}{18} & \frac{2}{9} & 0 \\ \frac{i}{18} & 0 & 0 & \frac{5}{18} \end{bmatrix}$$

Eigenvalues of $\tilde{\rho}_{12} : \frac{1}{6}, \frac{2}{9}, \frac{5}{18}, \frac{1}{3}$

Number of measurements needed to determine the eigenvalues of $\tilde{\rho}_{12}$

$$W = |\phi\rangle\langle\phi| - \frac{2}{9} I = \frac{1}{4}\left[\begin{array}{l} \frac{7}{9} I \otimes I + (\alpha^2 - \beta^2)\,(I \otimes \sigma_z + \sigma_z \otimes I) + \sigma_z \otimes \sigma_z \\ +2\alpha\beta\,(\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y) \end{array}\right]$$

$$\tilde{W} = (1-p)\,W + \frac{p}{4} I \quad \tilde{W} \to \text{Approximate Entanglement Witness } (AEW)$$

Choose minimum value of p i.e. $p_{\min}$ in such a way that $\tilde{W} \geq 0$.

$$Tr(W\tilde{\rho}_{12}) = \frac{1}{1-p_{\min}}\left[Tr(\tilde{W}\tilde{\rho}_{12}) - \frac{p_{\min}}{4}\right]$$

For our case, $p_{\min} = \frac{8}{15}$

$$Tr(\tilde{W}\tilde{\rho}_{12}) = F_{ave}(\tilde{W}, \tilde{\rho}_{12})$$

C. J. Kwong et.al., arXiv:1606.00427

C.J.Kwong et.al. also presented an experimental set up with single Hong-Ou-Mandel interferometry in which only two detectors are applied to calculate the average fidelity, regardless of the dimensions of the Hilbert space of the state.